JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 59, 119–129 (1977)

Series Inversion of Some Convolution Transforms

RICHARD ASKEY

Mathematics Research Center, University of Wisconsin, Madison, Wisconsin 53706

AND

DEBORAH TEPPER HAIMO

University of Missouri at St. Louis, St. Louis, Missouri 63121

Submitted by R. P. Boas

1. INTRODUCTION

Particularly simple algorithms have been derived for the inversion of some integral transforms when the functions represented by them also have a series expansion. In [6], Widder generalized results obtained earlier for some specific transforms by developing an inversion theory for the integral transform

$$f(x) = \int_0^\infty K(x/y)\,(\phi(y)/y)\,dy \tag{1.1}$$

when the generating function f has a series expansion

$$f(x) = \sum_{k=0}^\infty a_k x^k. \tag{1.2}$$

He established the result for the kernel $K(x)$ defined by

$$1/E(s) = \int_0^\infty x^{s-1} K(x)\,dx, \tag{1.3}$$

where the inversion function E belongs to one of two subclasses of the Laguerre–Polya class of entire functions. The subclasses considered include the class A_1 consisting of those functions E for which

$$E(s) = \prod_{k=1}^\infty (1 - (s^2/a_k^2)), \tag{1.4}$$

where

$$0 < a_1 \leqslant a_2 \leqslant \cdots \tag{1.5}$$

ISSN 0022-247X

and for some positive numbers Ω and b,

$$a_k \geqslant (k/\Omega) - b, \qquad k = 1, 2, 3, \ldots, \tag{1.6}$$

and the class A_2 comprising those functions E for which

$$E(s) = e^{-cs} \prod_{k=1}^{\infty} (1 - (s/a_k)) \, e^{s/a_k}, \tag{1.7}$$

where (1.5) and (1.6) hold, and, in addition,

$$\sum_{k=1}^{\infty} (1/a_k) = \infty. \tag{1.8}$$

Under rather general conditions, Widder proved that if (1.1), (1.2), and (1.3) hold with E in either class A_1 or A_2, then

$$\phi(x) = \sum_{k=1}^{\infty} a_k E(-k) \, x^k. \tag{1.9}$$

The case of the heat transform

$$f(t) = 2 \int_0^{\infty} (1/(4\pi t)^{1/2}) \, e^{-y^2/4t} \Phi(y) \, dy \tag{1.10}$$

treated in [5] and so called because of the fact that its kernel $(1/(4\pi t)^{1/2}) \, e^{-x^2/4t}$ is the source solution of the classical heat equation

$$(\partial^2/\partial x^2) \, u(x, t) = (\partial/\partial t) \, u(x, t); \tag{1.11}$$

its generalization, the reduced Poisson–Hankel transform considered in [2] and given by

$$f(t) = \int_0^{\infty} G(y; t) \, \Phi(y) \, d\mu(y), \tag{1.12}$$

where

$$G(y; t) = (1/2t)^{\nu + (1/2)} \, e^{-y^2/4t}, \qquad \nu > 0, \tag{1.13}$$

$$d\mu(y) = (2^{\nu - (1/2)} \Gamma(\nu + \tfrac{1}{2}))^{-1} \, y^{2\nu} \, dy; \tag{1.14}$$

and the potential transform [4] given by

$$f(x) = \int_0^{\infty} (y/(x^2 + y^2)) \, \Phi(y) \, dy, \tag{1.15}$$

which owes its name to its relationship to the Poisson integral representation of a function harmonic in the half-plane, all are convolution transforms of the form (1.1) with $K(x)$ defined by (1.3) where E belongs to A_1 or A_2. On the other hand, the Hankel potential transform studied in [1],

$$f(x) = \int_0^\infty (y/(x^2 + y^2)^{\nu+1}) \, \Phi(y) \, d\mu(y), \tag{1.16}$$

is not convered by the general Widder inversion algorithm. It can be, however, if the class A_1 is enlarged to include functions E with both positive and negative zeros symmetric about a point not necessarily the origin. Basically the same techniques can be extended to apply to the larger class. It is our goal to establish the inversion formula for this class and to illustrate the result with some of the many examples which satisfy the new conditions but not the old ones.

2. Preliminaries

The inversion algorithm to be derived applies when the inversion function belongs to a class A defined as follows.

Definition 2.1. E belongs to class A if and only if

$$E(s) = \prod_{k=1}^{\infty} \left(1 - \frac{s}{a_k}\right)\left(1 + \frac{s}{a_k + \epsilon}\right), \tag{2.1}$$

where

$$a_1 + \epsilon > 0 \qquad \text{and} \qquad 0 < a_1 \leqslant a_2 \leqslant \cdots, \tag{2.2}$$

and for some positive number Ω and b

$$a_k \geqslant (k/\Omega) - b, \qquad k = 1, 2, \ldots. \tag{2.3}$$

We then define the inversion operator as follows.

Definition 2.2. If

$$\theta = -xD, \tag{2.4}$$

where D denotes differentiation with respect to x, then

$$L_{n,x}[f]\,(x) = \prod_{k=1}^{n} \left(1 - \frac{\theta}{a_k}\right)\left(1 + \frac{\theta}{a_k + \epsilon}\right) f(x) \tag{2.5}$$

and

$$E(\theta)\,[f]\,(x) = \lim_{n\to\infty} L_{n,x}[f]; \tag{2.6}$$

we note that

$$L_{n,x}[x^\alpha] = \prod_{k=1}^{n} \left(1 + \frac{\alpha}{a_k}\right) \left(1 - \frac{\alpha}{a_k + \epsilon}\right) x^\alpha \tag{2.7}$$

so that $L_{n,x}$ is a differential operator which annihilates the functions x^{-a_k}, $x^{(a_k+\epsilon)}$.

The inversion of the transform (1.1) by the operator $E(\theta)$ is well known and is given in [3], where the following result has been established.

THEOREM 2.3. *If*

$$f(x) = \int_0^\infty K(x/u)\,(\phi(u)/u)\,du \tag{2.8}$$

with the integral converging for some $x > 0$, *and if the inversion function* E *where*

$$1/E(s) = \int_0^\infty x^{s-1} K(x)\,dx \tag{2.9}$$

belongs to class A, *then for almost all* $x > 0$,

$$E(\theta) f(x) = \phi(x). \tag{2.10}$$

We will appeal to this basic result in deriving the series inversion when f is represented not only by the transform (1.1) but by a power series as well. We need first an estimate of the inversion function. To this end, we take note of the following elementary inequality.

LEMMA 2.4. *If* α, β *are any positive constants, then for all* $x \geqslant 0$,

$$x^\alpha \leqslant K e^{\beta x}, \tag{2.11}$$

where K *is a positive constant depending on* α *and* β.

We now establish the asymptotic behavior of the inversion function.

LEMMA 2.5. *If* $E \in A$, *then for some positive numbers* K *and* δ,

$$E(-(\epsilon/2) + ix) \leqslant K e^{(\pi\Omega+\delta)x}, \tag{2.12}$$

where Ω *is the constant of* (2.3).

Proof. We have

$$\begin{aligned} E\left(-\frac{\epsilon}{2} + ix\right) &= \prod_{k=1}^{\infty} \left(1 - \frac{-(\epsilon/2) + ix}{a_k}\right) \left(1 + \frac{-(\epsilon/2) + ix}{a_k + \epsilon}\right) \\ &= \prod_{k=1}^{\infty} \frac{1 + (x/(a_k + \epsilon/2))^2}{1 - ((\epsilon/2)/(a_k + \epsilon/2))^2} \end{aligned} \tag{2.13}$$

so that

$$\log E\left(-\frac{\epsilon}{2}+ix\right)=\sum_{k=1}^{\infty}\log\frac{1+(x/(a_k+\epsilon/2))^2}{1-((\epsilon/2)/(a_k+\epsilon/2))^2}. \tag{2.14}$$

Let us now define a function N by

$$\begin{aligned} N(t) &= 0, \qquad 0 \leqslant t < a_1+\epsilon/2, \\ &= k, \qquad a_k+\epsilon/2 \leqslant t < a_{k+1}+\epsilon/2, \quad k=1,2,..., \end{aligned} \tag{2.15}$$

so that

$$\begin{aligned} \log E\left(-\frac{\epsilon}{2}+ix\right) &= \int_0^\infty \log\frac{1+(x^2/t^2)}{1-((\epsilon/2)^2/t^2)}\,dN(t) \\ &= \int_0^\infty \log\left(1+\frac{x^2}{t^2}\right)dN(t)+c \\ &= 2x^2\int_0^\infty \frac{N(t)}{t(x^2+t^2)}\,dt+c, \end{aligned} \tag{2.16}$$

where the last integral on the right is a result of an integration by parts and where c is a constant depending on ϵ. Now, since by (2.3),

$$N(t) \leqslant (t+b-(\epsilon/2))\,\Omega, \tag{2.17}$$

we have

$$\begin{aligned} \log E\left(-\frac{\epsilon}{2}+ix\right) &\leqslant 2x^2\Omega\int_0^\infty \frac{dt}{x^2+t^2}+2x^2\Omega\left(b-\frac{\epsilon}{2}\right)\int_{a_1+(\epsilon/2)}^\infty \frac{dt}{t(x^2+t^2)}+c \\ &= \pi\Omega x+\Omega\left(b-\frac{\epsilon}{2}\right)\log\left(1+\frac{x^2}{(a_1+\epsilon/2)^2}\right)+c. \end{aligned}$$

Hence, appealing to Lemma 2.4, we have

$$E(-(\epsilon/2)+ix) \leqslant Ke^{(\pi\Omega+\delta)x}.$$

LEMMA 2.6. *If*

$$S_n(x)=\prod_{k=1}^{n}\left(1-\frac{x}{a_k}\right)\left(1+\frac{x}{a_k+\epsilon}\right),$$

where the a_k *satisfy* (2.2), (2.3) *for* $k=1,2,...,$ *then there exist positive numbers* M, γ, x_0 *such that*

$$S_n(-(\epsilon/2)+ix) \leqslant Me^{\gamma x}, \qquad x>x_0, \quad n=1,2,....$$

Proof. We have

$$S_n\left(-\frac{\epsilon}{2}+ix\right)=\prod_{k=1}^{n}\left(1-\frac{-(\epsilon/2)+ix}{a_k}\right)\left(1+\frac{-(\epsilon/2)+ix}{a_k+\epsilon}\right)$$

$$=\prod_{k=1}^{n}\frac{1+(x/(a_k+\epsilon/2))^2}{1-((\epsilon/2)/(a_k+\epsilon/2))^2}$$

and since, clearly,

$$\left(1+\left(\frac{x}{a_k+\epsilon/2}\right)^2\right)\Big/\left(1-\left(\frac{\epsilon/2}{a_k+\epsilon/2}\right)^2\right)>1,$$

$S_n(-(\epsilon/2)+ix)$ is an increasing function of n. Hence

$$0<S_n(-(\epsilon/2)+ix)<E(-(\epsilon/2)+ix)$$

and the result follows by the preceding lemma.

3. Series Inversion Algorithm

When a function represented by the convolution transform whose kernel belongs to a restricted Laguerre–Polya class also has a series expansion, then the inversion of the transform is effected simply by the introduction of suitable multipliers into the series. The following results establish the inversion formulas.

Theorem 3.1. *If*

$$f(x)=\int_0^\infty K(x/u)\,(\phi(u)/u)\,du,\qquad 0<x<\infty, \tag{3.1}$$

where the inversion function E *given by*

$$1/E(s)=\int_0^\infty x^{s-1}K(x)\,dx \tag{3.2}$$

belongs to class A, *and if, for some real* α, ρ,

$$f(x)=\sum_{k=0}^{\infty}c_kx^{k+\alpha},\qquad 0<x<\rho, \tag{3.3}$$

then for almost all x *in some interval* $0<x<\rho_1$,

$$\phi(x)=\sum_{k=0}^{\infty}c_kE(-k-\alpha)\,x^{k+\alpha}. \tag{3.4}$$

Proof. Since a power series may be differentiated termwise, we have, on appeal to (2.7),

$$L_{n,x}f(x) = \sum_{k=0}^{\infty} c_k S_n(-k-\alpha)\, x^{k+\alpha}, \qquad 0 < x < \rho, \tag{3.5}$$

where

$$S_n(x) = \prod_{k=1}^{n} \left(1 - \frac{x}{a_k}\right)\left(1 + \frac{x}{a_k + \epsilon}\right). \tag{3.6}$$

We now must establish that we may take the limit as $n \to \infty$ within the summation. To this end, for fixed $x > 0$, we have

$$\left| \sum_{k=0}^{\infty} c_k S_n(-k-\alpha)\, x^{k+\alpha} \right| \leqslant \sum_{k=0}^{\infty} |\, c_k \,|\, |\, S_n(-k-\alpha)|\, x^{k+\alpha}.$$

But

$$\left| S_n\left(-x - \frac{\epsilon}{2}\right) \right| = \prod_{k=1}^{n} \left| \frac{(a_k + (\epsilon/2))^2 - x^2}{a_k(a_k + \epsilon)} \right| \leqslant S_n\left(-\frac{\epsilon}{2} + ix\right)$$

so that, involving Lemma 2.6, we find that

$$\left| \sum_{k=0}^{\infty} c_k S_n(-k-\alpha)\, x^{k+\alpha} \right| \leqslant M \sum_{k=0}^{\infty} |\, c_k \,|\, e^{\gamma(k+\alpha-(\epsilon/2))} x^{k+\alpha}.$$

Since the dominant series is independent of n and converges for $0 < x < \rho_1 = \rho e^{-\gamma}$, we have

$$E(\theta)\,[f]\,(x) = \sum_{k=0}^{\infty} c_k E(-k-\alpha)\, x^{k+\alpha}, \qquad 0 < x < \rho_1 .$$

But, for almost all $x > 0$, $E(\theta)\,[f] = \phi$ by Theorem 2.3, and hence the result.

An analogous theorem holds when F has a series expansion in negative powers of x.

THEOREM 3.2. *If $f(x)$ is given by* (3.1) *with* (3.2) *holding, and if, in addition,*

$$f(x) = \sum_{k=0}^{\infty} (c_k/x^{k+\alpha}), \qquad \rho < x < \infty, \tag{3.7}$$

then, for almost all x, $\rho_1 < x < \infty$,

$$\phi(x) = \sum_{k=0}^{\infty} c_k E(k+\alpha)\, x^{-(k+\alpha)}. \tag{3.8}$$

Proof. We have, in this case, upon differentiating termwise as before,

$$L_{n,x}f(x) = \sum_{k=0}^{\infty} c_k S_n(k+\alpha)\, x^{-(k+\alpha)}, \qquad \rho < x < \infty.$$

Further,

$$\left| \sum_{k=0}^{\infty} c_k S_n(k+\alpha)\, x^{-(k+\alpha)} \right| \leqslant \sum_{k=0}^{\infty} |\, c_k \,|\, e^{\nu(k+\alpha)} x^{-(k+\alpha)},$$

where the dominant series converges for $x > \rho_1 = e^{\pi\Omega}\rho$ and hence the proof follows as in the preceding theorem.

From the fact that by (2.7) and (3.5)

$$L_{n,x}(x^{\alpha}) = S_n(-\alpha)\, x^{\alpha} \tag{3.9}$$

so that, on differentiating with respect to α, we have

$$L_{n,x}(x^{\alpha} \log x) = -S_n'(-\alpha)\, x^{\alpha} + S_n(-\alpha)\, x^{\alpha} \log x, \tag{3.10}$$

the inversion algorithm applies when f itself does not have a series expansion but when f divided by the logarithmic function is expressible in a power series. We then have the following result.

THEOREM 3.3. *If $f(x)$ is given by* (3.1) *with* (3.2) *holding, and if, in addition,*

$$f(x) = \log x \sum_{k=0}^{\infty} c_k x^{k+\alpha}, \qquad 0 < x < \rho, \tag{3.11}$$

then, for almost all x, $0 < x < \rho_1$,

$$\phi(x) = \log x \sum_{k=0}^{\infty} c_k E(-k-\alpha)\, x^{k+\alpha} - \sum_{k=0}^{\infty} c_k E'(-k-\alpha)\, x^{k+\alpha}. \tag{3.12}$$

4. EXAMPLES

If we choose

$$K(x) = \frac{r\Gamma(q)}{\Gamma(p/r)\,\Gamma((rq-p)/r)} \frac{x^{p}}{(1+x^{r})^{q}} \tag{4.1}$$

then

$$\begin{aligned} 1/E(s) &= \int_0^{\infty} x^{s-1} K(x)\, dx \\ &= \frac{r\Gamma(q)}{\Gamma(p/r)\,\Gamma((rq-p)/r)} \int_0^{\infty} \frac{x^{s+p-1}}{(1+x^{r})^{q}}\, dx, \qquad 0 < s+p < rq, \\ &= \Gamma\left(\frac{p+s}{r}\right) \Gamma\left(\frac{rq-p-s}{r}\right) \Big/ \Gamma\left(\frac{p}{r}\right) \Gamma\left(\frac{rq-p}{r}\right) \end{aligned}$$

or

$$E(s) = \prod_{k=0}^{\infty} \left(1 + \frac{s}{p + rk}\right) \left(1 - \frac{s}{rq - p + rk}\right)$$

$$= \Gamma\left(\frac{p}{r}\right) \Gamma\left(\frac{rq - p}{r}\right) \Big/ \Gamma\left(\frac{p + s}{r}\right) \left(\frac{rq - p - s}{r}\right) \tag{4.2}$$

so that, clearly, $E(s)$ belongs to class A. If $p = 2\nu + 1$, $r = 2$, $q = \nu + 1$, then

$$f(x) = \int_0^\infty K(x/t) (\phi(t)/t) \, dt \tag{4.3}$$

becomes the Hankel potential transforms [1],

$$F(x) = \int_0^\infty \frac{t}{(x^2 + t^2)^{\nu+1}} \Phi(t) \, d\mu(t), \qquad d\mu(t) = \frac{t^{2\nu} \, dt}{2^{\nu+(1/2)} \Gamma(\nu + \frac{1}{2})}, \tag{4.4}$$

where

$$\begin{aligned} F(x) &= x^{-2\nu-1} f(x), \\ \Phi(t) &= t^{-2\nu-1} \phi(t), \end{aligned} \tag{4.5}$$

and Theorems 3.1 and 3.2 become Theorems 5.1 and 5.2 in [1]. Various examples to illustrate Theorems 3.1 and 3.2 in this case are given in [1]. As another particular example of (4.3), consider

$$f(x) = \frac{2\Gamma(\nu + 1) \, x^{2\nu+1}}{\pi^{1/2} \Gamma(\nu + \frac{1}{2})} \int_0^\infty \frac{t^{2\nu}}{(x^2 + t^2)^{\nu+1}} \mathscr{J}(t) \, dt, \tag{4.6}$$

where

$$\mathscr{J}(t) = 2^{\nu-(1/2)} \Gamma(\nu + \tfrac{1}{2}) \, t^{(1/2)-\nu} J_{\nu-(1/2)}(t), \tag{4.7}$$

$J_\alpha(t)$ being the ordinary Bessel function of order α. But the integral of (4.7) can be evaluated explicitly so that

$$\begin{aligned} f(x) &= x^{2\nu} e^{-x} \\ &= \sum_{k=0}^{\infty} ((-1)^k/k!) \, x^{k+2\nu}, \qquad 0 < x < \infty. \end{aligned} \tag{4.8}$$

On the other hand,

$$E(-k - 2\nu) = \pi^{1/2} \Gamma(\nu + \tfrac{1}{2}) \Big/ \Gamma\left(\frac{1 - k}{2}\right) \Gamma\left(\frac{1 + k + 2\nu}{2}\right) \tag{4.9}$$

so that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} E(-k-2\nu)\, x^{k+2\nu} = \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma(\nu+\frac{1}{2})}{2^{2k}k!\,\Gamma(\nu+\frac{1}{2}+k)}\, x^{2k+2\nu} \tag{4.10}$$
$$= x^{2\nu}\mathscr{J}(x), \qquad 0 < x < \infty,$$
$$= \phi(x)$$

as predicted by Theorem 3.1.

Caution must be exercised in the application of the algorithms as the following example illustrates.

Let

$$J_0(x) = \frac{2\Gamma(\nu+1)\, x^{2\nu+1}}{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})} \int_0^\infty \frac{t^{2\nu}}{(x^2+t^2)^{\nu+1}}\, \Phi(t)\, dt. \tag{4.11}$$

As above,

$$E(s) = \Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Big/\Gamma\left(\nu+\frac{1}{2}+\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right),$$

so that

$$E(-2k) = \frac{\Gamma(k+\frac{1}{2}-\nu)\,\Gamma(\frac{1}{2})\,(-1)^k}{\Gamma(\frac{1}{2}-\nu)\,\Gamma(k+\frac{1}{2})},$$

and a formal application of Theorem 3.1 gives

$$\Phi(k) = t^{-2\nu} \sum_{k=0}^{\infty} \frac{(-1)^k\, E(-2k)\, t^{2k}}{k!k!2^{2k}}$$
$$= t^{-2\nu} \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2}-\nu)} \frac{\Gamma(\frac{1}{2})}{\Gamma(k+\frac{1}{2})} \frac{(t/2)^{2k}}{k!k!}.$$

But this is clearly wrong, since the kernel, $K(x) = x^{2\nu+1}/(x^2+1)^{\nu+1}$ is positive, $\Phi(t)$ is positive, whereas $J_0(x)$ changes sign. The problem arises because $\Phi(t)$ is not integrable with respect to $t^{2\nu}/(x^2+t^2)^{\nu+1}$ for any x, so the integral equation does not have a solution.

The transform related to ultraspherical polynomials and given by

$$f(x) = \int_0^1 \frac{1-r^2}{(1-2xr+r^2)^{\nu+1}}\, \phi(r)\, dr \tag{4.12}$$

is a Hankel potential transform, as can be verified by the change of variables

$$r = 1 + y^2 - y(2+y^2)^{1/2}.$$

The algorithm derived can thus be applied to invert (4.12) when f has a series representation.

References

1. F. M. Cholewinski and D. T. Haimo, Inversion of the Hankel potential transform, *Pacific J.* **37** (1971), 319–331.
2. F. M. Cholewinski and D. T. Haimo, Inversion of the reduced Poisson–Hankel transforms, *J. Analyse Math.* **25** (1972), 323–343.
3. I. I. Hirschman and D. V. Widder, "The Convolution Transform," Princeton Univ. Press, Princeton, N.J., 1955.
4. D. V. Widder, A transform related to the Poisson integral for a half plane, *Duke Math. J.* **33** (1966), 355–362.
5. D. V. Widder, Inversion of a heat transform by use of series, *J. Analyse Math.* **18** (1967), 389–413.
6. D. V. Widder, Inversion of a convolution transform by use of series, *J. Analyse Math.* **21** (1968), 293–312.